(Spatial interpretation of some Canonical Transformations).

Rest-frame rotations of spin $\frac{1}{2}$ Particles.

1. Introduction

~~The extension of classes of possible interactions among them brought about by non-conservation of parity and charge conjugation in weak interactions has revived interest in the study of canonical transformations involving charge conjugate and parity conjugate ($\gamma_5\psi$) functions. —~~

2. The Λ group

~~The transformation~~ Let the canonical ~~transformation~~ be represented ~~$\psi \to e^{i\gamma_5 a}\psi$~~ by the unitary matrix S, so that the 4-spinor wave function ψ undergoes the transformation

$$\psi \to S\psi . \qquad (1)$$

The particular transformation

$$S_0 = e^{i\gamma_5 a} = \cos a + i\gamma_5 \sin a \quad (a = \text{real}) \qquad (2)$$

where we are using the representation in which $\gamma_5^2 = 1$, has been ~~studied~~ used by the author(), Touschek[1], Nishijima[2] and others. Pauli[3] has recently introduced the transformation

$$S_1\psi = \psi\cos b + \gamma_5\psi^c \sin b \quad (b = \text{real}) \qquad (3)$$

where

$$\psi^c = C\psi^*$$

is the charge conjugate function. The charge conjugation operator Γ may be written as

$$\Gamma = CK \qquad (4)$$

where K is the operation of complex conjugation and C is a matrix such that $\gamma_\mu^T = -C\gamma_\mu C^{-1}$

In the following we use the representation for γ_μ where γ_2 is real, so that we can take $C = \gamma_2$.

Hence the ~~matrix~~ operator corresponding to the canonical transformation (3) may be written as

$$S_2 = e^{\gamma_5 \cancel{\gamma_4} \Gamma b} = e^{\gamma_5 C K b}. \qquad (5)$$

Of course we also have the gauge transformation ~~which we write as~~ is represented by

$$S_3 = e^{i\omega} \qquad (\omega \text{: real}). \qquad (6)$$

We now define the operators:

$$\cancel{T_1 = -i\gamma_5\gamma_4\Gamma = -i\gamma_5 C K \quad T_2 = \quad T_3 = \gamma_5}$$

$$T_3 = \gamma_5 \qquad (\;)\, i\alpha_3$$

$$T_1 = -i\gamma_5\Gamma = -i\gamma_5 C K \qquad (\;)\, i\alpha_1$$

$$T_2 = \cancel{\gamma_3 T_1} = +\Gamma = +CK \qquad (\;)\, i\alpha_2$$

and $T_4 = -i\gamma_5$

which have the properties

$T_1 T_4 = T_4 T_1$, $\quad T_1^2 = T_2^2 = T_3^2 = 1$, $\quad T_1 T_2 T_3 = T_4$, $\quad T_4^2 = -1$

and $T_1 T_2 = -T_2 T_1 = i T_3$ ~~etc~~ and cycl. per.

~~T_1, T_2 and T_3 all commute with $T_4 = i\gamma_5$.~~

~~and~~ Thus T_1, T_2, T_3 and T_4 are ~~seen to be~~ isomorphic to the Pauli spin matrices and i respectively. The three unitary operators ~~matrices~~ which can be defined by means of them corresponding to 3-dimensional rotation matrices are

$$S_n = e^{i\gamma_5 T_n a_n} \qquad (n = 1, 2, 3)$$

where the parameters a_n are real and no summation is implied over n).

~~It is then possible to give a representation~~

~~Representation~~ ?

We note the important fact that the six operators T_n and $i\gamma_5 T_n$ commute with the three Dirac operators $\alpha_n = \gamma_0 \gamma_n$.

$$\Big(\underbrace{P\tfrac{1+i\gamma_5}{2} + N i\varepsilon_1 \tfrac{1-i\gamma_5}{2}}_{\chi}\Big)\Big(\underbrace{\tfrac{1-i\gamma_5}{2} P^+ - \tfrac{1+i\gamma_5}{2} i\varepsilon_1 N^+}_{\zeta^+}\Big)$$

$$= -P\tfrac{1+i\gamma_5}{2} i\varepsilon_1 N^+ + N i\varepsilon_1 \tfrac{1-i\gamma_5}{2} P^+$$

$$\zeta\,\chi^+$$

$$\Big(P\tfrac{1-i\gamma_5}{2} - N i\varepsilon_1 \tfrac{1+i\gamma_5}{2}\Big)\Big(\tfrac{1+i\gamma_5}{2} P^+ + \tfrac{1-i\gamma_5}{2} i\varepsilon_1 N^+\Big)$$

$$= P\tfrac{1-i\gamma_5}{2} i\varepsilon_1 N^+ - N\tfrac{1-i\gamma_5}{2} i\varepsilon_1 P^+$$

$$\chi\zeta^+ = N\tfrac{i\varepsilon_1 - \varepsilon_2}{2} P^+ - P\tfrac{i\varepsilon_1 - \varepsilon_2}{2} N^+$$

$$\zeta\chi^+ = P\tfrac{i\varepsilon_1 + \varepsilon_2}{2} N^+ - N\tfrac{i\varepsilon_1 + \varepsilon_2}{2} P^+$$

$$-N\tfrac{\varepsilon_2}{\gamma_5} P^+ + P\varepsilon_2 N^+$$

$$N i\varepsilon_1 P^+ - P i\varepsilon_1 N^+$$

$$-N i\varepsilon_1 \tfrac{1-i\gamma_5}{2} i\varepsilon_1 N^+$$

$$\chi\tfrac{1+i\gamma_5}{2} = P\tfrac{1+i\gamma_5}{2}$$

$$\zeta\tfrac{1+i\gamma_5}{2} = -N i\varepsilon_1 \tfrac{1+i\gamma_5}{2}$$

$$\chi\tfrac{1+i\gamma_5}{2}\chi^+ = P\tfrac{1+i\gamma_5}{2} P^+$$

$$\zeta\tfrac{1+i\gamma_5}{2}\zeta^+ = -N i\varepsilon_1 \tfrac{1+i\gamma_5}{2} i\varepsilon_1 N^+$$

We now consider the six parameter group of transformations which we call internal Lorentz transformations in contrast with the ordinary (or external) Lorentz transformations

$$\Lambda: \quad \psi'' = \Lambda\psi \Rightarrow e^{\vec{\lambda}\cdot\vec{T}} e^{\vec{\mu}\cdot T_4\vec{T}} \psi$$

In 2×2 matrix representation we have

$$L: \quad \Psi \to M\Psi$$

and $$\Lambda: \quad \Psi \to \Psi N$$

where M and N are unimodular matrices,
When L and Λ are combined we have

$$\Psi \to M\Psi N \text{ which preserves } |\Psi|^2.$$

3. The ~~neutrino case~~ Dirac equation

Now we prove the following theorem.

The ~~neutrino equation~~ Lorentz invariant neutrino equation (Dirac equation for a particle with rest mass zero) is also invariant with respect to Λ transformations.

This follows from the fact that the Dirac equation for $m = 0$ can be written as

$$(\partial_0 + \vec{\alpha}\cdot\vec{\nabla})\psi_\nu = 0$$

and that $\vec{T}$ and $T_4\vec{T}$ commute with $\vec{\alpha}$. This implies that the neutrino, unlike the electron has a new kind of 4-dimensional symmetry.

In the electron case we have

$$(\partial_0 + \vec{\alpha}\cdot\vec{\nabla})\psi = m\beta\psi.$$

which, on applying the Λ transformation becomes

$$(\partial_0 + \vec{\alpha}''\cdot\vec{\nabla})\psi'' = m(\Lambda\beta\Lambda^{-1})\psi'' = m\beta''\psi''$$

~~Th~~ give because $\vec{\alpha}'' = \Lambda\vec{\alpha}\Lambda^{-1} = \vec{\alpha}$. This ~~canonical~~ transformation

Actually, it is possible to find a representation in which the correspondence between the spin matrices and the T operators are explicitly displayed can be found ~~its~~ in the following way. ~~We write the 2 two spinors ξ_i and η_i~~ Weyl ( ) has shown that Dirac's equation can be written in 2 spinor form by means of ~~the~~ 2-spinors ξ_i and η_i $(i=1,2)$. If now we define ~~the~~ a 2×2 matrix Ψ by

$$\Psi = \begin{pmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{pmatrix}$$

then, using the relations

$$\xi_1 = \psi_1 + \psi_3 \qquad \eta_1 = -\psi_2^* + \psi_4^*$$
$$\xi_2 = \psi_2 + \psi_4 \qquad \eta_2 = \psi_1^* - \psi_3^*$$

which hold between the Dirac 4-spinor ψ and the Weyl 2-spinors ξ and η, it can be shown easily that we have the following correspondences:

$$\sigma_n \Psi \longleftrightarrow \alpha_n \psi = \gamma_0 \gamma_n \psi$$
$$(\bar{\Psi})^+ = \sigma_2 \Psi \sigma_2 \longleftrightarrow \beta \psi = \gamma_0 \psi$$
$$\Psi \sigma_n \longleftrightarrow T_n \psi$$
$$i\Psi \longleftrightarrow T_4 \psi$$

where $\bar{\Psi}$ means the adjoint matrix and the dagger denotes hermitian conjugation.

These correspondences ~~$\longleftrightarrow \beta\psi$~~ have already been ~~cited~~ established ~~also~~ and used ~~also in previous articles by the author~~ elsewhere ( ).

Under a Lorentz transformation, the transformation law for ψ is

$$L: \quad \psi' = L\psi = e^{\frac{1}{2} a_{\mu\nu} \sigma_{\mu\nu}} \psi = e^{\cancel{\frac{1}{2}(\vec{a}\cdot\vec{\sigma} + \vec{b}\cdot\beta\vec{\sigma})}} \psi$$

where $\sigma_{\mu\nu} = \frac{1}{2}(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu)$ and the parameters $a_{\mu\nu}$ are real. This can be written as $\psi' = e^{\frac{1}{2}\vec{a}\cdot\vec{\sigma}} e^{\vec{b}\cdot\beta\vec{\sigma}} \psi$ ~~$\vec{\sigma}$ is Pauli vector, and~~ The components of $\vec{\sigma}$ are the Pauli matrices doubled and $\vec{b}$ represents the direction and magnitude of spatial rotation while $\vec{a}$ defines a pure Lorentz transformation.

does not change the commutation relations of matrices γ_μ. ~~since $\vec{\alpha}$ and β are changed~~
But it is not canonical because ~~it affe~~ one can show that the commutation relations are in general affected by it.
~~We find~~ If invariance of the rest mass term is also postulated we find

$$\Lambda = e^{\cancel{T_1} T_2 \lambda}\, e^{T_4 T_3 \cancel{\beta} \gamma^\mu} = e^{T_2 \lambda} e^{i\gamma^\mu}.$$

This is ~~the~~ gauge transformation followed by a linear combination of the wave function and its charge conjugate, the ~~final~~ transformed wave function having same normalization as the initial one.

<u>Interpretation of the Λ transformations as rest-frame Lorentz transformations in the rest frame of the fermion.</u>

<u>4 - Definition of the rest frame of the ~~fermion~~ electron.</u>

~~We now~~ To obtain a better insight into the nature of the Λ transformations we first have to define what we mean by the rest frame of a Dirac particle - Let ψ be a plane wave solution of the Dirac equation normalized so that $\bar{\psi}\psi = 1$.

We have the following 4-vectors

$$J_\mu = \bar{\psi}\gamma_\mu\psi$$

$$S_\mu = \bar{\psi}\gamma_5\gamma_\mu\psi$$

which denote respectively the ~~spin and~~ 4-current and the spin density 4-vectors - We can also define the complex 4-vector $B_\mu = \bar{\psi}\gamma_\mu\psi^c = B^{(1)}_\mu + i B^{(2)}_\mu$
where $B^{(1)}$ and $B^{(2)}$ denote respectively the real and the imaginary part of B_μ